BIBLIOGRAPHY

1. Riazin, V. A., Optimal one-time correction in a model problem. Teoriia Veroiatnostei i ee Primenenie, Vol. 11, №4, 1966.
2. Shelement'ev, G. S., Optimal combination of control and tracking. PMM Vol. 32, №2, 1968.
3. Chernous'ko, F. L., A minimax problem of one-time correction with measurment errors. PMM Vol. 32, №4, 1968.
4. Shelement'ev, G. S., On a certain motion correction problem. PMM Vol. 33. №2, 1969.
5. Krasovskii, N. N., The Theory of Motion Control. Moscow, "Nauka", 1968.
6. Krasovskii, N. N., On a differential convergence game. Dokl. Akad. Nauk SSSR, Vol. 182, №6, 1968.
7. Pontriagin, L. S., Boltianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., The Mathematical Theory of Optimal Processes. Moscow, Fizmatgiz, 1961.

Translated by A. Y.

ONE FORM OF THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS

PMM Vol. 33, №3, 1969, pp. 397-402
FAM GUEN
(Hanoi and Moscow)
(Received November 18, 1968)

Displacement operators constructed with the aid of all the constraints are used to derive a form of the equations of motion which is valid for both holonomic and nonholonomic mechanical systems. In the case of holonomic systems the equations coincide with the familiar equations of Poincaré [1, 2].

1. Constructing the displacement operators. Let the positions of a mechanical system with l degrees of freedom be defined by the n variables $x_1, \ldots, x_n$ subject to $n - l$ linear constraints

$$\eta_j\, dt \equiv \sum_{i=1}^{n} a_{ji}\, dx_i + a_{j0}\, dt = 0 \qquad (j = l+1, \ldots, n) \tag{1.1}$$

on the real displacenets, and to the equations

$$\omega_j \equiv \sum_{i=1}^{n} a_{ji} \delta x_i = 0 \qquad (j = l+1, \ldots, n) \tag{1.2}$$

on the virtual displacements.

Here a_{ji}, a_{j0} are functions of the variables t, x_i; dx_i, δx_i are the differentials and variations of the variables x_i on the real and virtual displacements of the system.

Following Chetaev [2], we complement (1.2) by a system of l linear differential forms

$$\omega_1, \ldots, \omega_l \tag{1.3}$$

which are independent of each other and also with respect to the forms $\omega_{l+1}, \ldots, \omega_n$ of (1.2). Next, we define the total variation of the function $f(t, x_i)$ by the formula

$$\delta f = \sum_{j=1}^{n} \omega_j X_j (f), \qquad X_j = \sum_{i=1}^{n} \xi_j^i \frac{\partial}{\partial x_i} \qquad (j = 1, \ldots, n) \tag{1.4}$$

Here ξ_j^i are definite functions of the variables t, x_i which depend on the choice of forms (1.3).

By virtue of (1.2) and (1.4) the change (variation) of the function f on the virtual displacements of the system is

$$\delta f = \sum_{s=1}^{l} \omega_s X_s (f) \tag{1.5}$$

The symbols $X_1, \ldots, X_l$ are called the "operators", and forms (1.3) the "parameters" of the virtual displacements of the system.

Similarly, we complement (1.1) by the forms

$$\eta_1 dt, \ldots, \eta_l dt, dt \tag{1.6}$$

which are linear, independent both of each other and of forms (1.1), and such that the total differential of the function $f(t, x_i)$ is given by the formula

$$df = dt \Big[X_0 (f) + \sum_{j=1}^{n} \eta_j X_j (f)\Big] \qquad \Big(X_0 = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \xi_0^i \frac{\partial}{\partial x_i}\Big) \tag{1.7}$$

for which the X_j are operators (1.4).

Bearing in mind (1.1), we obtain the change in the function f on the real displacements of the system in the form

$$df = dt \Big[X_0 (f) + \sum_{s=1}^{l} \eta_s X_s (f)\Big] \tag{1.8}$$

The symbols $X_0, X_1, \ldots, X_l$ are called the "operators" of the real displacements of the system, and $\eta_1, \ldots, \eta_l$ their "parameters". Here ξ_0^i are definite functions of the variables t, x_i which depend on the choice of (1.6).

We can show that the system of virtual-displacement operators of a system is closed if the mechanical system is holonomic, and that otherwise it is open.

In fact, since the outer derivative of total variation (1.4) equals zero,

$$0 \equiv (\delta f)' = \sum_{j=1}^{n} \omega_j' X_j (f) + \sum_{(i, j)} [\omega_i, \omega_j] (X_i, X_j) f \tag{1.9}$$

and since the outer derivatives ω_t' for forms (1.2), (1.3) can be written in the form

$$\omega_t' = - \sum_{(i, j)} C_{ijt} [\omega_i, \omega_j] \qquad (t = 1, \ldots, n) \tag{1.10}$$

it follows from (1.9) that [3]

$$(X_i, X_j) = \sum_{t=1}^{n} C_{ijt} X_t \qquad (i, j = 1, \ldots, n) \tag{1.11}$$

Here C_{ijt} are functions of the variables t, x_i. Without limiting generality we can assume that the last $n - k$ constraints of (1.2) are holonomic ($l \leqslant k \leqslant n$).

Then, by the Frobenius theorem [3], we must have

$$C_{ijt} = 0 \qquad (i, j = 1, \ldots, n; \ t = k + 1, \ldots, n) \tag{1.12}$$

in (1.10), and that the commutators of the virtual displacement operators are given by

$$(X_r, X_s) = \sum_{t=1}^{l} C_{rst} X_t + \sum_{\nu=l+1}^{k} C_{rs\nu} X_\nu \qquad (r, s = 1, \ldots, l) \tag{1.13}$$

In the case of holonomic systems all the constraints (1.2) are holonomic, $k = l$, $C_{rs\nu}=0$, so that, by (1.13), the operators $X_1,\ldots, X_l$ form a closed system [3]. In the case of nonholonomic systems, when the first $k - l$ constraints of (1.2) do not form a completely integrable system together with the remaining constraints, it follows by the Frobenius theorem that the coefficients $C_{rs\nu}$ in (1.13) cannot all equal zero, so that the system of virtual displacement operators for nonholonomic systems cannot (by definition) be closed.

By a similar argument we obtain the commutators of the real displacement operators of the system under consideration in the form

$$(X_r, X_s) = \sum_{t=1}^{l} C_{rst} X_t + \sum_{\nu=l+1}^{k} C_{rs\nu} X_\nu \qquad \begin{pmatrix} r = 0, 1, \ldots, l \\ s = 1, \ldots, l \end{pmatrix} \tag{1.14}$$

where all of the $C_{rs\nu}$ equal zero if the system is holonomic. Otherwise it is impossible for all these coefficients to equal zero.

Using the terminology of the theory of groups of continuous transformations in the space of the variables x_i on constraints (1.2), we note that $X_1,\ldots, X_l$ constitute the operators of infinitely small transformation (1.6) with the parameters $\omega_1,\ldots,\omega_l$ which transform a point with the coordinates x_i to the neighboring point $x_i + \delta x_i$ along constraints (1.2). If all of constraints (1.2) are holonomic, and if the coefficients C_{rst} in (1.13) are constant ($C_{rs\nu} = 0$), the above operators form a Lie group [1, 2, 3]. If the system is nonholonomic and C_{rst} and $C_{rs\nu}$ are constant, we have an incomplete Lie group whose operators do not form a closed system.

2. The equations of motion. Let all the constraints of a mechanical system be ideal, and let the active forces have the force function U. Substituting the expressions for the virtual displacements of the system points defined by (1.6),

$$\delta u_i = \sum_{s=1}^{l} \omega_s X_s(u_i), \qquad \delta v_i = \sum_{s=1}^{l} \omega_s X_s(v_i), \qquad \delta w_i = \sum_{s=1}^{l} \omega_s X_s(w_i) \qquad (i = 1, \ldots, N) \tag{2.1}$$

into the general equation of dynamics, by virtue of the independence of the parameters $\omega_1, \ldots, \omega_l$, we obtain

$$\sum_{i=1}^{N} m_i [u_i'' X_s(u_i) + v_i'' X_s(v_i) + w_i'' X_s(w_i)] - X_s(U) = 0 \qquad (s = 1, \ldots, l) \tag{2.2}$$

or

$$\frac{d}{dt} \sum_{i=1}^{N} m_i [u_i' X_s(u_i) + v_i' X_s(v_i) + w_i' X_s(w_i)] - X_s(U) -$$
$$- \sum_{i=1}^{N} m_i \left[u_i' \frac{dX_s(u_i)}{dt} + v_i' \frac{dX_s(v_i)}{dt} + w_i' \frac{dX_s(w_i)}{dt}\right] = 0 \qquad (s = 1, \ldots, l) \tag{2.3}$$

Here N is the number of material points of the system; u_i, v_i, w_i are the Cartesian coordinates of the i th point of mass m_i; u_i'', v_i'', w_i'' are its accelerations; u_i', v_i', w_i' are the velocities given (according to (1.9)) by the formulas

$$u_i' = X_0(u_i) + \sum_{s=1}^{l} \eta_s X_s(u_i), \qquad v_i' = X_0(v_i) + \sum_{s=1}^{l} \eta_s X_s(v_i)$$
$$w_i' = X_0(w_i) + \sum_{s=1}^{l} \eta_s X_s(w_i) \qquad (i = 1, \ldots, N) \tag{2.4}$$

From (2.4) we obtain

$$X_s(u_i) = \frac{\partial u_i'}{\partial \eta_s}, \quad X_s(v_i) = \frac{\partial v_i'}{\partial \eta_s}, \quad X_s(w_i) = \frac{\partial w_i'}{\partial \eta_s} \qquad \begin{pmatrix} s = 1, \ldots, l \\ i = 1, \ldots, N \end{pmatrix} \tag{2.5}$$

and from (1.9) by virtue of (1.14) and (1.17),

$$\frac{dX_s(u_i)}{dt} = X_s(u_i') + \sum_{t=1}^{l} \Big(C_{0st} + \sum_{r=1}^{l} \eta_r C_{rst}\Big) X_t(u_i) +$$
$$+ \sum_{\nu=l+1}^{k} \Big(C_{0s\nu} + \sum_{r=1}^{l} \eta_r C_{rs\nu}\Big) X_\nu(u_i) \qquad \begin{pmatrix} s = 1, \ldots, l \\ i = 1, \ldots, N \end{pmatrix} \tag{2.6}$$

Formulas (2.6) for the coordinates v_i, w_i are obtainable in similar fashion.
By virtue of (2.5) and (2.6) we can transform Eqs. (2.6) into

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_s} - X_s(T + U) - \sum_{t=1}^{l} \Big(C_{0st} + \sum_{r=1}^{l} \eta_r C_{rst}\Big)\frac{\partial T}{\partial \eta_t} -$$
$$- \sum_{\nu=l+1}^{k} \Big(C_{0s\nu} + \sum_{r=1}^{l} \eta_r C_{rs\nu}\Big)\Big(\frac{\partial T^\circ}{\partial \eta_\nu}\Big) = 0 \qquad (s = 1, \ldots, l) \tag{2.7}$$

This is the required form of the equations of motion which is valid for both holonomic and nonholonomic systems. Here T is the kinetic energy of the system under consideration and the $(\partial T^\circ / \partial \eta_\nu)$ are given by

$$\Big(\frac{\partial T^\circ}{\partial \eta_\nu}\Big) = \sum_{i=1}^{N} m_i \left[u_i' X_\nu(u_i) + v_i' X_\nu(v_i) + w_i' X_\nu(w_i)\right] \qquad (\nu = l + 1, \ldots, k) \tag{2.8}$$

which have the mechanical significance of the momenta associated with the parameters η_ν for the so-called "associated" holonomic system with the kinetic energy T° obtainable from the system under consideration by omitting the first $k - l$ constraints from (1.1) and (1.2) [4].

Equations (2.7) coincide with the familiar equations of Poincaré [1–3] if the system is holonomic, since all of the coefficients $C_{rs\nu}$ in (2.7) are then equal to zero. If the system is nonholonomic, (2.7) are equivalent to Eqs. (1.13) of [4], since, taking l forms (1.7) and $k - l$ of the forms $\eta_{l+1}, \ldots, \eta_k$ of (1.1) as the parameters of the real displacements of the associated holonomic system, then Eqs. (1.13) of [4] assume the form (2.7).

The equivalence of Eqs. (2.7) to the equations of Appell [4],

$$\frac{\partial S}{\partial \eta_s'} = X_s(U) \qquad (s = 1, \ldots, l) \tag{2.9}$$

follows from the relations

$$\frac{\partial S}{\partial \eta_s'} = \frac{d}{dt}\frac{\partial T}{\partial \eta_s} - \sum_{i=1}^{N} m_i \left[u_i' \frac{dX_s(u_i)}{dt} + v_i' \frac{dX_s(v_i)}{dt} + w_i' \frac{dX_s(w_i)}{dt}\right] \quad (s = 1, \ldots, l) \tag{2.10}$$

whose right sides are (by (2.3)) transformable into (2.7). Here S is the acceleration energy of the system and $\eta_s' = d\eta_s / dt$.

When x_i are generalized Lagrange coordinates and constraints (1.1) have the special form

$$\eta_j \equiv x_j' - \sum_{s=1}^{l} a_{js} x_s' - a_{j0} = 0 \qquad (j = l + 1, \ldots, n) \tag{2.11}$$

Eqs. (2.7) become the equations of Hamel transformed to the kinetic energy T.

3. Examples. 1°. The equations of motion of a hoop. The positions of a hoop moving along a horizontal plane can be defined in terms of the parameters $\theta, \psi, \varphi, \xi, \eta, \zeta$ under the constraints [6]

$$\begin{gathered}\eta_4 \equiv \xi' - a\sin\theta\sin\psi\theta' + a\cos\theta\cos\psi\psi' + a\cos\psi\varphi' = 0\\ \eta_5 \equiv \eta' + a\sin\theta\cos\psi\theta' + a\cos\theta\sin\psi\psi' + a\sin\psi\varphi' = 0\\ \eta_6 \equiv \zeta' - a\cos\theta\theta' = 0\end{gathered} \tag{3.1}$$

Taking

$$\eta_1 = \theta', \quad \eta_2 = \psi'\sin\theta, \quad \eta_3 = \psi'\cos\theta + \varphi' \tag{3.2}$$

as the parameters of the real displacements of the hoop, we obtain

$$\begin{gathered}X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial\theta} + a\sin\theta\sin\psi\frac{\partial}{\partial\xi} - a\sin\theta\cos\psi\frac{\partial}{\partial\eta} + a\cos\theta\frac{\partial}{\partial\zeta}\\ X_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} - \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}, \quad X_3 = \frac{\partial}{\partial\varphi} - a\cos\psi\frac{\partial}{\partial\xi} - a\sin\psi\frac{\partial}{\partial\eta}\\ X_4 = \frac{\partial}{\partial\xi}, \quad X_5 = \frac{\partial}{\partial\eta}, \quad X_6 = \frac{\partial}{\partial\zeta}\end{gathered} \tag{3.3}$$

The first four of these operators are the operators of the real displacements of the hoop. Their commutators are

$$\begin{gathered}(X_0, X_1) = (X_0, X_2) = (X_0, X_3) = 0, \quad (X_1, X_2) = -(X_2, X_1) = -\operatorname{ctg}\theta X_2 + X_3\\ (X_1, X_3) = 0, \quad (X_2, X_3) = -(X_3, X_2) = \frac{a\sin\psi}{\sin\theta}X_4 - \frac{a\cos\psi}{\sin\theta}X_5\end{gathered} \tag{3.4}$$

For the hoop we have

$$\begin{gathered}T = {}^1\!/_2[(A + a^2)\eta_1^2 + A\eta_2^2 + (C + a^2)\eta_3^2], \qquad U = -ag\sin\theta\\ T^\circ = {}^1\!/_2(\eta_4^2 + \eta_5^2 + \eta_6^2 + 2a\sin\theta\sin\psi\,\eta_1\eta_4 - 2a\sin\theta\cdot\cos\psi\,\eta_1\eta_5 +\\ + 2a\cos\theta\,\eta_1\eta_6 - 2a\cos\psi\,\eta_3\eta_4 - 2a\sin\psi\,\eta_3\eta_5) + \ldots\end{gathered} \tag{3.5}$$

where the ellipsis represents terms not containing η_4, η_5, η_6.

Substituting (3.3)–(3.5) into (2.7), we obtain the equations of motion of a hoop [4, 6],

$$\begin{gathered}(A + a^2)\eta_1' - A\operatorname{ctg}\theta\eta_2^2 + (C + a^2)\eta_2\eta_3 + ag\cos\theta = 0\\ A\eta_2' + A\operatorname{ctg}\theta\,\eta_1\eta_2 - C\eta_1\eta_3 = 0, \quad (C + a^2)\eta_3' - a^2\eta_1\eta_2 = 0\end{gathered} \tag{3.6}$$

2°. A holonomic system of Appell. All of the constraints of the holonomic system considered by Appell in Sect. 469 of [6] are holonomic,

$$\eta_4 \equiv \xi' + a\sin\theta\theta' = 0, \quad \eta_5 \equiv \eta' = 0, \quad \eta_6 \equiv \zeta' - a\cos\theta\theta' = 0 \tag{3.7}$$

Again taking (3.2) as the parameters of the real displacements, we obtain

$$\begin{gathered}X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial\theta} - a\sin\theta\frac{\partial}{\partial\xi} + a\cos\theta\frac{\partial}{\partial\eta}, \quad X_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} - \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}\\ X_3 = \frac{\partial}{\partial\varphi}, \quad X_4 = \frac{\partial}{\partial\xi}, \quad X_5 = \frac{\partial}{\partial\eta}, \quad X_6 = \frac{\partial}{\partial\zeta}\end{gathered} \tag{3.8}$$

The real-displacement operators X_0, X_1, X_2, X_3 form a closed system, since

$$\begin{gathered}(X_0, X_1) = (X_0, X_2) = (X_0, X_3) = 0, \ (X_1, X_2) = -(X_2, X_1) = -\operatorname{ctg}\theta X_2 + X_3\\ (X_1, X_3) = -(X_3, X_1) = 0, \quad (X_2, X_3) = -(X_3, X_2) = 0\end{gathered} \tag{3.9}$$

The kinetic energy and the force function of the system are

$$\begin{gathered}T = {}^1\!/_2[(A + a^2)\eta_1^2 + A\eta_2^2 + (C + a^2)\eta_3^2], \qquad U = -ag\sin\theta\\ T^\circ = {}^1\!/_2(\eta_4^2 + \eta_5^2 + \eta_6^2 - 2a\sin\theta\eta_1\eta_4 + 2a\cos\theta\eta_1\eta_6 + \ldots\end{gathered} \tag{3.10}$$

where the ellipsis denotes terms not containing η_4, η_5, η_6. By virtue of (3.8)–(3.10) Eqs. (2.7) yield

$$(A + a^2)\eta_1' - A \operatorname{ctg} \theta \eta_2^2 + (C + a^2)\eta_2\eta_3 + ag\cos\theta = 0 \tag{3.11}$$
$$A\eta_2' + A \operatorname{ctg}\theta\eta_1\eta_2 - (C + a^2)\eta_1\eta_3 = 0, \quad (C + a^2)\eta_3' = 0$$

These are the equations of motion of the holonomic system of Appell. They differ from the equations of motion of a hoop (3.6), even though they share the same expression for the kinetic energy T. This qualitative difference is also noticeable in the fact that the displacement operators of the hoop do not form a closed system.

The above examples show once again that Eqs. (2.7) can be used without determining in advance whether the system is holonomic or nonholonomic and which of the constraints imposed on the system are nonholonomic (as is necessary, for example, in the case of Eqs. (1.13) of [4]).

3°. **The equations of motion of Chaplygin's sled on an inclined plane.** Let the plane on which the sled is moving form the angle θ with the horizontal plane, and let $O\xi\eta$ be some fixed coordinate system attached to this plane; $O\xi$ is the fast-descent axis and $O\eta$ is the horizontal axis. Defining the positions of the sled in terms of the coordinates ξ, η of the point of tangency A of the sled with the inclined plane and by means of the angle φ between $O\xi$ and the axis Ax directed along the plane of the wheel, we obtain the equation of the (nonholonomic) constraint in the form [7]

$$\eta_3 \equiv \xi'\sin\varphi - \eta'\cos\varphi = 0 \tag{3.12}$$

Let us take $\eta_1 = \varphi'$, $\eta_2 = \xi'\cos\varphi + \eta'\sin\varphi$ as the parameters of the real displacements of the sled. This yields (3.13)

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial\varphi}, \quad X_2 = \cos\varphi\frac{\partial}{\partial\xi} + \sin\varphi\frac{\partial}{\partial\eta}, \quad X_3 = \sin\varphi\frac{\partial}{\partial\xi} - \cos\varphi\frac{\partial}{\partial\eta}$$

The sled displacement operators X_0, X_1, X_2 form an incomplete Lie group since their commutators are

$$(X_0, X_1) = (X_0, X_2) = 0, \ (X_1, X_2) = -(X_2, X_1) = -X_3 \tag{3.14}$$

The kinetic energy T, T° and the force function U for the sled on an inclined plane are

$$T = \frac{m}{2}(\gamma^2\eta_1^2 + \eta_2^2 - 2\beta\eta_1\eta_2), \quad T^\circ = \frac{m}{2}(\eta_3^2 - 2\alpha\eta_1\eta_3) + \ldots \tag{3.15}$$

$$U = mg\sin\theta\,(\xi + \alpha\cos\varphi - \beta\sin\varphi)$$

The ellipsis in the expression for T° represents the terms not containing η_3. Substituting (3.13)–(3.15) into (2.7) and solving them for η_1', η_2', we obtain

$$\eta_1' = \frac{\alpha\eta_1}{\gamma^2 - \beta^2}(\beta\eta_1 - \eta_2) - \frac{\alpha g\sin\theta}{\gamma^2 - \beta^2}\sin\varphi$$
$$\eta_2' = \frac{\alpha\eta_1}{\gamma^2 - \beta^2}(\gamma^2\eta_1 - \beta\eta_2) + \frac{g\sin\theta}{\gamma^2 - \beta^2}[(\gamma^2 - \beta^2)\cos\varphi - \alpha\beta\sin\varphi] \tag{3.16}$$
$$\gamma^2 = \alpha^2 + \beta^2 + k^2$$

Here α, β are the coordinates of the center of mass of the sled in the moving system Axy attached to the sled; k is the radius of gyration of the sled about the center of mass. We note that a particular case of Eqs. (3.16) is considered in [8].

Equations (3.16) together with

$$\varphi' = \eta_1, \quad \xi' = \eta_2\cos\varphi, \quad \eta' = \eta_2\sin\varphi \tag{3.17}$$

determine η_1, η_2, φ, ξ, η as functions of the time t.

We note that if we take $\eta_1 = \varphi'$, $\eta_2 = \xi'$ as the real-displacement parameters of the sled, then the resulting displacement operators do not form an incomplete Lie group, since not all of the coefficients and their commutators are constant.

The author is grateful to V. V. Rumiantsev for his useful comments concerning the present study.

BIBLIOGRAPHY

1. Poincaré, M. H., Sur une forme nouvelle des équations de la mécanique. Compt. Rend., Vol. 132, 1901.
2. Chetaev, N. G., On the Poincaré equations. PMM Vol. 5, №2, 1941.
3. Cartan, É., Integral Invariants (Russian translation). Moscow-Leningrad, Gostekhteorizdat, 1940.
4. Fam Guen, On the equations of motion of nonholonomic mechanical systems in Poincaré-Chetaev variables. PMM Vol. 32, №5, 1968.
5. Hamel, G., Die Lagrange-Eulerischen Gleichungen der Mechanik. Z. Math. Phys. Vol. 50, 1904.
6. Appell, P., Theoretical Mechanics, (Russian translation) Vol. 2, Moscow, Fizmatgiz, 1960.
7. Chaplygin, S. A., On the theory of motion of nonholonomic systems. The reducing multiplier theorem. Mat. Sb. Vol. 28, №2, 1912.
8. Neimark, Iu. I and Fufaev, N. A., The Dynamics of Nonholonomic Systems. Moscow, "Nauka", 1967.

Translated by A. Y.

PERIODIC SOLUTIONS OF SYSTEMS WITH LAG CLOSELY RELATED TO LIAPUNOV SYSTEMS

PMM Vol. 33, №3, 1969, pp. 403-412

A. F. KLEIMENOV and S. N. SHIMANOV
(Sverdlovsk)

(Received December 8, 1968)

The familiar definition of Liapunov systems [1] is generalized for systems with lag. The present paper concerns a system closely related to that of Liapunov involving a small addition periodic in t. A theorem concerning the existence of a periodic solution is proved. An example is investigated.

1. Let us consider the system described by equations with lag of the form

$$\frac{dx}{dt} = \int_{-\tau}^{0} x(t+\vartheta)\, d\eta(\vartheta) + X(x(t+\vartheta)) + \mu F(t, x(t+\vartheta), \mu) \qquad (1.1)$$

where x is an n-dimensional vector and $\eta(\vartheta)$ is an $n \times n$ matrix of the functions $\eta_{ij}(\vartheta)$ with bounded variation defined on the segment $[-\tau, 0]$; the integral is to be interpreted in the Stieltjes sense; $X(x(\vartheta)) = \{X_i(x(\vartheta))\}$ is a nonlinear functional defined on the piecewise continuous functions $x(\vartheta)$, $-\tau \leqslant \vartheta \leqslant 0$ (with discontinuities of the first kind) bounded in norm, i.e. $\|x(\vartheta)\| < R$, where $R > 0$,

$$\|x(\vartheta)\| = \sup(|x_1(\vartheta)|, \ldots, |x_n(\vartheta)|), \quad -\tau \leqslant \vartheta \leqslant 0 \qquad (1.2)$$

Substituting any vector function $x(y, \vartheta)$ analytic in y and differentiable with respect to ϑ into the functional $X(x(\vartheta))$, we obtain the analytic function $X(x(y, \vartheta)) = X_1(y)$.